Mathematical Physics

Notes

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MATHMATICAL PHYSICS

- Differential equations
- Group Theory
- Complex Variable
- Matrix and tensors

UNIT-2
Group Theory

• Group –

A group is an algebric structure $(G, *)$ consisting of a set of elements equipped with a binary operation which combines any two elements to form a third element. To be a group, this operation must satisfy four following axioms.

1. Closure Axioms –

$$a, b \in G \Rightarrow a * b \in G \quad \forall a, b \in G$$

2. Associative Axioms –

$$a, b, c \in G \Rightarrow$$

$$\boxed{a * (b * c) = (a * b) * c}$$

3. Identity Axioms –

$$a, b \in G \Rightarrow$$

There exist an element $e \in G$ s.t.

$$\boxed{a * e = e = e * a}$$

4. Inverse Axiom –

For each element of G possesses inverse.

$$a, b \in G \Rightarrow$$

$$a * b = b * a = e$$

And

$$b = a^{-1}$$

Then

$$\boxed{a * a^{-1} = a^{-1} * a = e}$$

If any groupic algebric structure $(G, *)$ satisfies these five conditions, it will be a group.

5. Commutative group –

$$a, b \in G$$

$$\boxed{a * b = b * a \in G}$$

so it is called an abelian group.

- History of Group theory-

The original motivation for group theory was the quest for solution of polynomial equations of degree higher than 4. In 19th century Evariste Galois gave a criterion (मापदंड) for the solvability of a particular polynomial equation in terms of the symmetrical group of its roots. The elements of such a galois group corresponding a certain permutations of the roots.

- Subgroup-

A subgroup H of a group G is a non-empty subset of G that forms a group under the binary operation of G. Equivalenty, H is a non-empty subset of G s.t. If $a, b \in H$, so does ab^{-1}.

$$1 = aa^{-1} \in H$$

Also

$$ab = a(b^{-1})^{-1} \in H$$

→ Properties of subgroup.

(i) The Intersection of two subgroup is a subgroup of G.
(ii) The union of two subgroup is not necessary a subgroup.

→ Necessary and sufficient condition-

For a subgroup H of a group G to be a subgroup:

(i) $a \in H, b \in H \Rightarrow a * b \in H$
(ii) $a \in H, a^{-1} \in H$
(iii) $a \in H, b \in H \Rightarrow a \cdot (b^{-1})^{-1} \in H$

- Cyclic group-

A group G is cyclic group if $G = \langle a \rangle$ for some $a \in G$ and $\langle a \rangle$

$$\langle a \rangle = \{ a^k \mid k \in \mathbb{Z} \}$$

Such an element is called a generator of G.

OR

In Mathematics, a cyclic group is a group that is generated by a single element.

- Homomorphism-

Let $(G,*)$ and $(G',*')$ are groups then a mapping $f: g \longrightarrow g'$ is said to be homomorphism if

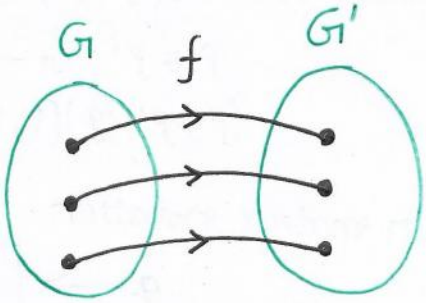
$$f(a*b) = f(a)*'f(b)$$

where $a,b \in G$ and $f(a), f(b) \in G'$

- Isomorphism-

Let $(G,*)$ and $(G',*')$ are two groups. Then a mapping $f: G \longrightarrow G'$ is called isomorphism if

(i) f is one-one.
(ii) f is onto.
(iii) f is homomorphism
$f(a*b) = f(a)*'f(b)$ $\forall\, a,b \in G$



→ $G = \{1, \omega, \omega^2\}$ of order 3 under multiplication.

$*$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

- Representation Theory of finite group-

Let G be a finite group. Let k be a field. We will assume that characteristic of k is 0 e.g. $\mathbb{C}, \mathbb{R}$ or $\mathbb{Q}$ though often $\text{char}(K) \nmid |G|$ is enough.

A representation of G over k is a homomorphism,

$$\rho: G \longrightarrow GL(V)$$

where V is a vector space of finite dimension over field k. The vector space V is called a representation space of G and its dimension the dimension of representation.

- Reducible Representation –

A representation (ρ, V) is called completely reducible if it is a direct sum of irreducible ones. Equivalently if $V = W * \dots * W_r$, where each W_i is G-invariant Irreducible representations.

- Irreducible Representation –

A representation (ρ, V) of G is called irreducible if it has no proper invariant subspace i.e. only invariant subspaces are 0 and V.

Let (ρ, V) and (ρ', V') be two representations of G over field k.

$$\rho * \rho' : G \longrightarrow GL(V * V')$$
$$(\rho * \rho')(g)(v, v') = (\rho)(g)(v), \rho'(g)(v')$$

In matrix notation

$$g \longrightarrow \begin{bmatrix} \rho(g) & 0 \\ 0 & \rho'(g) \end{bmatrix}$$

This motivates us to look at those nice representations which can be obtained by taking direct sum of irreducible ones.

- Schur's Lemma Theorem –

Definition – Let (ρ, V) and (ρ', V') be two representations of G over field k. A linear map

$$T : V \longrightarrow V'$$

is called a G-map if it satisfies the following;

$$\rho'(t)\, T = T[\rho(t)] \quad \forall\, t \in G$$

(G-map)

- Statement –

Let (ρ, V) and (ρ', V') be two irreducible representations of G (of dimension ≥ 1)

Let $T : V \longrightarrow V'$ be a G-map.

Then

$T = 0$ or T is isomorphism.

Moreover

If T is nonzero

Then T is isomorphism.

($\because$ If and only If two representations are equivalent.)

Proof -

Let us consider the subspace $Ker(T)$. We claim that it is a G-invariant subspace of V.

$$v \in Ker(T)$$

Then

$$T\rho(t)(v) = \rho'(t)T(v) = 0$$

$$\rho(t)(v) \in Ker(T) \quad \forall\, t \in G$$

Now applying Maschke's Theorem on irreducible representation V. we get,

$$Ker(T) = 0 \quad \text{or} \quad Ker(T) = V$$

Now assume $Ker(T) = 0$

$\therefore$ T is injective

Now we consider

$$Im(T) \subset V'$$

And It is also a G-varient.

Let $y = T(x) \in Im(T)$

$$\rho'(t)(y) = \rho'(t)T(x) = T\rho(t)(x) \in Im(T) \quad \forall\, t \in G$$

Hence Im is G-invarient

Again, applying Maschke's Theorem on irreducible representation V'. we get,

$$Im(T) = 0 \quad \text{or} \quad Im(T) = V'$$

T is Injective.

$Im(T) \neq 0$

Hence

$Im(T) = V'$

It proves that,

T is an isomorphism.

Orthogonality Relations-

The Schur Orthogonality relation, which proves by Issai schur through schur's Lemma, express a central fact about representation of finite groups. They admit that a generalization to the case of compact groups in general and in particular compact Lie groups, such as the rotation group $(SO)_3$.

$$\langle \alpha, \beta \rangle := \frac{1}{|G|} \alpha(g)\overline{\beta(g)}$$

↑ complex conjugate

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 0 & \text{If } i \neq j \\ 1 & \text{If } i = j \end{cases}$$

$$\sum_{\chi_i} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |C_G(g)| \text{ , If } g, h \text{ are conjugate} \\ 0 \text{ , otherwise} \end{cases}$$

Statement- The set of irreducible character $\{\chi_1, \chi_2, \ldots\}$ form an orthonormal set of $(C(G), \langle , \rangle)$ That is,

(1) If χ is a character of an irreducible representation then $\langle \chi, \chi \rangle = 1$

(2) If χ and χ' are two irreducible characters of non isomorphic representation then $\langle \chi, \chi \rangle = 0$

Proof-

Let ρ and ρ' be non isomorphic irreducible representations. $\langle a_{ij} \rangle$ and $\langle b_{ij} \rangle$ be the corresponding matrix elements.

Then

$$\chi(g) = \Sigma_i \, a_{ii}(g)$$

$$\chi'(g) = \Sigma_j \, b_{jj}(g)$$

Then

$$\langle \chi, \chi' \rangle = \langle \chi, \chi' \rangle = \left(\Sigma_i \, a_{ii}, \Sigma_j \, b_{jj} \right) = \left(\Sigma_{ij} \left(a_{ii}, b_{jj} \right) \right) = 0$$

If $\frac{1}{|G|} \sum_{t \in G} a_{il}(t) \, a_{mj}(t^{-1}) = \frac{1}{n} \delta_{lm} \delta_{ij}$ and $\forall f \in C(G)$ then $(f, \chi) = \langle f, \chi \rangle$

Then

$$\langle \chi, \chi \rangle = \langle \chi, \chi \rangle = \left(\Sigma_i \, a_{ii}, \Sigma_j \, b_{jj} \right) = \Sigma_i (a_{ii}, b_{jj}) = \sum_{i=1}^{n} \frac{1}{n} = 1$$

↑ dimension of representation ρ.

- Character of Representation-

Let (ρ, V) be a representation of G. The character of (corresponding to) representation ρ is a map $\chi : G \longrightarrow C$ defined by $\chi(t) = tr(\rho(t))$

↑ trace of corresponding matrix.

Strictly speaking it is χ_ρ but for the simplicity of notation, we write χ only when it is clear which representation it corresponds to.

- Class-

In mathematics, especially group Theory, the elements of any group may be partitioned into conjugacy classes; members of the same conjugacy class share many properties, and study of conjugacy classes of non abelian groups reveals many Important features of their structure. For an abelian group, each conjugacy class is a set containing one element.

$$cl(a) = \{b \in G \mid \text{there exists } g \in G \text{ with } b = gag^{-1}\}$$

and is called the conjugacy class of a.

UNIT-1
Differential Equations

- Besell's Equation-

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

OR

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0 \quad \ldots\ldots\ldots (1)$$

Let us consider that eqⁿ solution of Besellel's diffⁿ equation on asending power of x.

$$y=x^K(a_0+a_1x+a_2x^2+\ldots\ldots+a_rx^r)$$

$$y=\sum_{r=0}^{\infty}a_rx^{K+r}$$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}(K+r)a_rx^{K+r-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(K+r)(K+r-1)x^{K+r-2}$$

substitute $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y values in eqⁿ (1)

$$\sum_{r=0}^{\infty}a_0(K+r)(K+r-1)x^{K+r-2}+\frac{1}{x}\sum_{r=0}^{\infty}a_r(K+r)x^{K+r-1}+\left(1-\frac{n^2}{x^2}\right)\sum_{r=0}^{\infty}a_rx^{K+r}=0$$

$$\sum_{r=0}^{\infty}a_r[(K+r)(K+r)-n^2]x^{K+r-2}+\sum_{r=0}^{\infty}a_rx^{K+r}=0 \quad \ldots\ldots\ldots (2)$$

This eqⁿ is identical & we can evaluated the diffⁿ power of x in eqⁿ(2). Let us first start with the smallest power i.e. x^{K+r-2}.

$$\sum_{r=0}^{\infty}a_r[(K+r)(K+r)-n^2]=0 \quad \ldots\ldots\ldots (3)$$

Putting $r=0$

$$a_0[(K)(K)-n^2]=0$$

$$a_0[K^2-n^2]=0$$

$$[K^2-n^2]=0$$

$$K^2 = n^2$$

$$\boxed{K = \pm n}$$

Putting $r = 1$

$$a_1[(K+1)^2 - n^2] = 0$$

$$(K+1)^2 = n^2 \neq 0$$

Then

$$\boxed{a_1 = 0}$$

$r \longrightarrow r+2$

Putting in eqn (2)

Then $$\sum_{r=0}^{\infty} a_{r+2}[(K+r+2)^2 - n^2]\, x^{K+r} + \sum_{r=0}^{\infty} a_{r+2} x^{K+r+2} = 0$$

Now comparing,

$$a_{r+2} = \frac{a_2}{(K+r+2)^2 - n^2}$$

$$= \frac{a_r}{(K+r+2+n)(K+r+2n)}$$

$r = 1, 3, 5 \ldots\ldots a_5$

→CASE I :

When $K = n$

$$a_{r+2} = \frac{-a_r}{(2n+r+2)(r+2)}$$

Now substituting $r = 0, 2, 4, 6, \ldots$

$r = 0$, then

$$a_2 = \frac{-a_0}{2(2n+2)} = \frac{-a_0}{(2)^2(n+1)}$$

$r = 2$, then,

$$a_4 = \frac{-a_2}{4(2n+4)} = \frac{a_0}{2 \cdot 4(2n+2)(2n+4)}$$

$r = 4$, then,

$$a_6 = \frac{-a_4}{(2)^4 \cdot 2!(2n+1)(2n+2)} = \frac{-a_0}{2 \cdot 4 \cdot 6(2n+2)(2n+4)(2n+6)}$$

Now,

$$y = x^n \left[a_0 + \frac{a}{2^2(n+1)} x^2 + \frac{a_0}{2^4 2!(n+1)(n+2)} x^4 + \dots \right]$$

$$y = x^n a_0 \left[1 + \frac{(-1)^1}{2^2 1!(n+1)} x^2 + \frac{(-1)^2}{2^4 2!(n+1)(n+2)} x^4 + \dots \right]$$

$$y = x^n a_0 \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{2r} r! [(n+1)(n+2)\dots(n+r)]}$$

If $a_0 = \frac{1}{2^n \lfloor n+1}$

The above eqn is known as Bessel's differential eqn solution of (1) kind where a_0 is constant

$$J_n(x) = x^n \frac{1}{2^n \lfloor n+1} \sum_{r=0}^{\infty} \frac{(-1)^r . x^{2r}}{2^{2r} r! (n+1)(n+2)\dots(n+r)}$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \lfloor n+r+1} \left(\frac{x}{2}\right)^{n+2r} \left\{ (n+1)\lfloor n+1 - \lfloor n+r+1 \right\}$$

This is required solution of this is 1st kind solution.

→ Case II :

$$K = -n$$

Then

$$J_r(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r \lfloor (-n)(r)+1} \left(\frac{x}{2}\right)^{-n+2r}$$

The second above solution is known as Beselle eqn of second kind.

Then

$$y = J_n(x) + J_{-n}(x)$$

• Beta and Gamma Function-

$\gamma \rightarrow$ Lower case gamma

$\Gamma \rightarrow$ Upper case gamma (In grik)

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$n > 0$

$$\Gamma n+1 = \int_0^\infty \underbrace{e^{-x} x^n}_{\text{By parts}} dx = \Pi(n)$$

$$\Pi(n) = \Gamma n+1$$

$$\Pi(n) = \frac{e^{-x}}{-1} x^n + \int e^{-x} \frac{d}{dx}(x^n)$$

$$\Pi(n) = \Gamma n+1 = -e^{-x} x^n + \int e^{-x} . n x^{n-1} dx$$

$$\Pi(n) = -e^{-x}[x^n]_0^\infty + n \int e^{-x} x^{n-1} dx$$

$$\Pi(n) = 0 + n \Gamma n$$

$$\boxed{\Gamma n+1 = n \Gamma n = \Pi(n)}$$

β = Lower case beta

B = Upper case beta

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \ldots\ldots\ldots (1)$$

$$B(m+1, n+1) = \int_0^1 x^m (1-x)^n dx$$

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{n+1}} dx \ldots\ldots\ldots (2)$$

Put $x = \sin^2\theta$

$dx = 2 \sin\theta \cos\theta \, d\theta$

$$\begin{bmatrix} x & 0 & 1 \\ \theta & 0 & \pi/2 \end{bmatrix}$$

$$B(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta \, d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta \quad \ldots\ldots (3)$$

Properties;

(i) $B(m,n) = B(n,m)$

(ii) $B(m,n) = \dfrac{\lceil m \lceil n}{\lceil m+n}$

- Poisson Equations-

Study this unit from classical Mechanics section

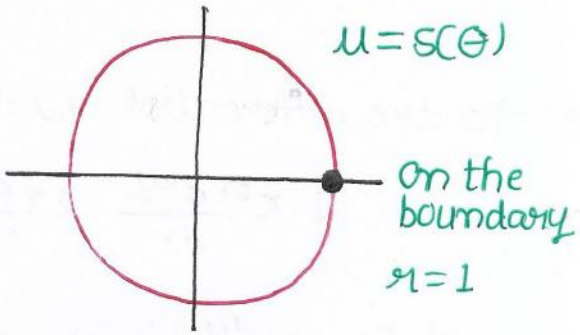
- Fourier Series of Laplace equation-

$$u(r,\theta) = \sum_{n=0}^{\infty} a_n r^n \cos n\theta + \sum_{n=0}^{\infty} b_n r^n \sin n\theta$$

$u(1,\theta)$ = delta function $\delta(\theta)$

$$\sum_{n=0}^{\infty} a_n \cos n\theta + \sum_{n=0}^{\infty} b_n \sin n\theta = \delta(\theta)$$

$b_n \sin n\theta$ → odd function cancelled



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\theta)\,d\theta = \frac{1}{2\pi}$$

$$a_n = \frac{1}{\pi}\int \delta(\theta)\cos n\theta \, d\theta = \frac{1}{\pi}$$

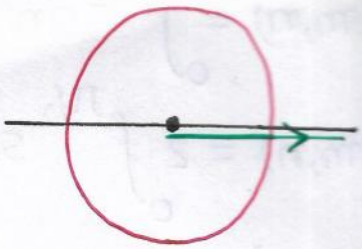
$$u(r,\theta) = \sum_{n=0}^{\infty} \frac{1}{2\pi} + \frac{1}{\pi}\cos n\theta \cdot r^n$$

$$u(r,\theta) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{r^n}{\pi}\cos n\theta = \frac{1}{2\pi}\cdot\frac{1-r^2}{1+r^2-2r\cos\theta}$$

Ray $\theta = 0$, $\cos\theta = 1$

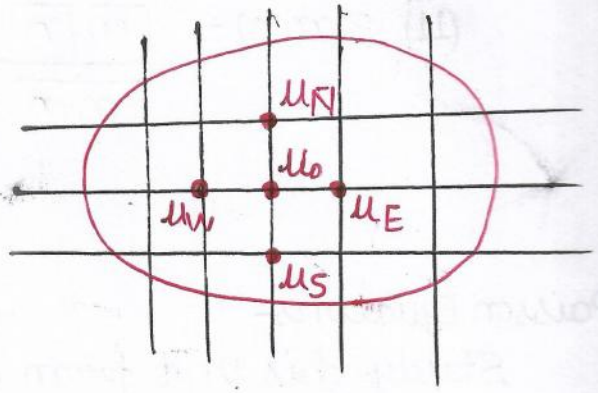
$$u(r,\theta) = \frac{1}{2\pi} \cdot \frac{1-r^2}{(1-r)^2}$$

$$\boxed{u(r,\theta) = \frac{1}{2\pi} \cdot \frac{1+r}{1-r}}$$



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\rightarrow u_0 = \frac{1}{4}\left(u_E + u_W + u_N + u_S\right)$$



- Legendre differential Equation-

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$\Rightarrow \frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

$$[\because n = 0, 1, 2, 3, \ldots.]$$

$x = \pm 1$ are the singular points and $x = 0$ is the ordinary point of the differential equation.

Properties of Legendre Polynomial:

- $P_n(1) = 1$
- $P_n(-1) = (-1)^n$
- $P_n(x)$ = even function if n is even
- $P_n(x)$ = odd function if n is odd

- Hermite differential equation–

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

$$[\because n = 0, 1 \ldots .]$$

$x = 0$ is an ordinary point of the Hermite differential eqⁿ.

Solution;

The solution of the Hermite differential eqⁿ is a terminating series, so it is called Hermite polynomial of order n i.e. $H_n(x)$

$$H_n(x) = \sum_{r=0}^{P} \frac{(-1)^r n! (2x)^{n-2r}}{r!(n-2r)!}$$

$$\Rightarrow P = \frac{n}{2}$$

if $n = 2, 4, 6, 8, \ldots .$

$$\Rightarrow P = \frac{n-1}{2}$$

if $n = 1, 3, 5, 7, 9, \ldots .$

- Lagauerre Equation of differenciation–

$$x\frac{d^2y}{d} + (1-x)\frac{dy}{dx} + ny = 0$$

$x = 0$ is a regular singular point of Laguerre differential eqⁿ.

Solution;

The solution

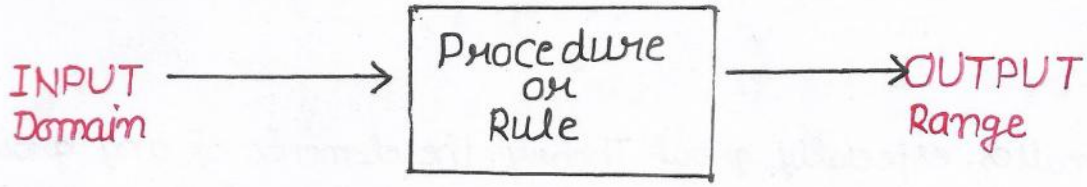
$$L_n(x) = \sum \frac{(-1)^r n! x^r}{(r!)^2 (n-r)!}$$

The solution of Lagauerre differential eqn is known as Lagauerre polynomial of order n i.e. $L_n(x)$.

UNIT-3
Complex Variables

- Functions of complex Variables-

A function is a procedure which gives a unique output for any suitable inputs is called the domain of the function while the set of outputs which are possible is called the range of the function.



e.g. Squareroot; the square of the output is equal to the input.

INPUT 4, 9 $\longrightarrow \sqrt{\quad} \longrightarrow$ OUTPUT 2, 3

$Z = x+iy \longrightarrow$ Function Machine $\longrightarrow w = f(z) = u+iv$

- Analytic Function-

In mathematical Physics, an analytic function is a function that totally given by a convergent series. There exist both real and complex analytic Function.
A function $f(z)$ is said to be analytic function if it is defined and differenciable at each point of a Region R. This function is called regular or holomorphic function.

Mathematically,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where f is real analytic Function
$x_0 \in D$
a_n is real number

$$T(x) = \sum_{n=0}^{\infty} \frac{(f^n)(x_0)}{n!} (x - x_0)^n$$

↑ Infinite diff^n funt.^{n}

• Cauchy's Integral Theorem-

In mathematics, the cauchy's Integral Theorem (formally known as the Cauchy-Goursat Theorem) in complex analysis, named after Augustin-Louis Cauchy is an important statement about line integrals for holomorphic function in complex plane.

↑ Function of one or more complex variables

→ Statement- If $f(z)$ is an analytic function of z and if $f'(z)$ is continuous at each point (of) within, an, close contour c then

$$\int_C f(z)\,dz = 0$$

→ Proof: Let $f(z)$ is a function of complex variables.

$$f(z) = u + iv$$
$$z = x + iy$$
$$dz = dx + i\,dy$$

Then

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C (u\,dx + iu\,dy + iv\,dx - v\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

($u\,dx$: P, $-v\,dy$: Q, $v\,dx$: P, $u\,dy$: Q)

• Green's Theorem

$$P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_C f(z)\,dz = \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$\int_C f(z)\,dz = \iint_S \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

- CR eq^n's

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= \frac{-\partial v}{\partial x} \end{aligned}\right\} \text{Apply in } \int_C f(z)\,dz$$

$$\int_C f(z)\,dz = \iint_S \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_S \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$\boxed{\int_C f(z)\,dz = 0}$$

This is desired result.

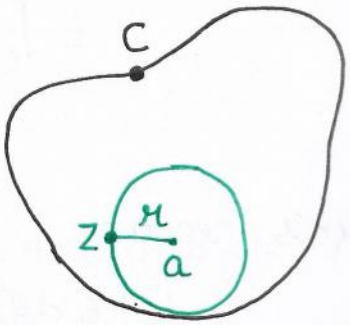
- Cauchy's Integral Formula-

In mathematics, cauchy's Integral Formula, named after Augustin-Louis Cauchy, is a cauchy's central statement in complex analysis. It expresses the fact that a holomorphic function defined on a disk is completely determined by its value of boundary of the disk.

→Statement→

If $f(z)$ is analytic & single valued within and on close contour 'C' and 'a' is any point on 'C' then,

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$



→ Proof; Let $z=a$ in a closed contour C. consider a circle r with center at $(z=a)$ and radius ρ.

consider the function

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz$$

—— Integral Theorem

$$|z-a| = r$$
$$z-a = re^{i\theta}$$
$$z = a+re^{i\theta} \quad \ldots\ldots(1)$$
$$dz = rie^{i\theta}\,d\theta \quad \ldots\ldots(2)$$

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, rie^{i\theta}\,d\theta$$

$$\oint_C \frac{f(z)}{z-a}\,dz = i\oint_{C_1} f(a+re^{i\theta})\,d\theta$$

$r \ll\ll\ll\ll\ll a$

Then

$$\oint_C \frac{f(z)}{z-a}\,dz = i\oint_{C_1} f(a)\,d\theta$$

$$\oint_C \frac{f(z)}{z-a}\,dz = if(a)\oint_{C_1} d\theta = if(a)\left[\theta\right]_0^{2\pi} = 2\pi i f(a)$$

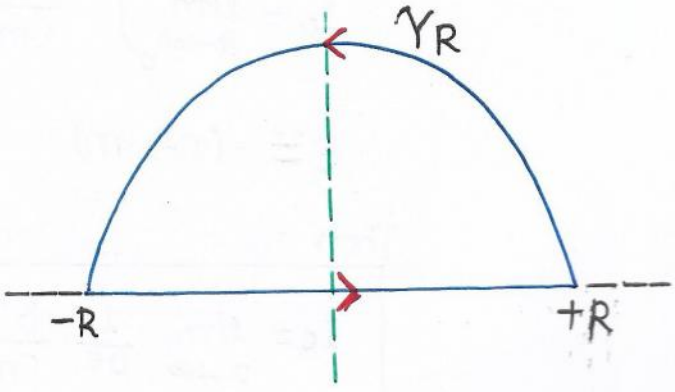
$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\boxed{f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz}$$

This is desired expression.

- Contour Integration-

Contour Integration is the process of calculating the value of a contour Integration around a given contour in a complex plane. As a result of a truly amazing property of



holomorphic function, such integrals can be computed easily simply by summing the value of the complex residue inside the contour.

Let $P(x)$ and $Q(x)$ be polynomials of polynomial degree n and m with cofficients $b_n \ldots\ldots b_0$ and $c_m, \ldots\ldots c_0$. Take that contour in the upper half plane, replace x by z and $z = Re^{i\theta}$ Then

$$\int_{-\infty}^{\infty} \frac{P(z)}{Q(z)}\,dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{P(z)}{Q(z)}\,dz$$

$$\lim_{R\to\infty} \int_{\gamma R} \frac{P(z)}{Q(z)}\,dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{P(Re^{i\theta})}{Q(Re^{i\theta})}\, iRe^{i\theta}\,d\theta$$

$$= 2\pi i \sum_{I\,|z|>0} \text{Res} \left| \frac{P(z)}{Q(z)} \right|$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{P(z)}{Q(z)}\,dz = 2\pi i \sum_{I|z|>0} \text{Res}\left[\frac{P(z)}{Q(z)}\right] - \lim_{R\to\infty} \int_{0}^{\pi} \frac{P(Re^{i\theta})}{Q(Re^{i\theta})}\, iRe^{i\theta}\,d\theta$$

$$I_R = \lim_{R\to\infty} \int_0^{\pi} \frac{P(Re^{i\theta})}{Q(Re^{i\theta})}\,d\theta \cdot iRe^{i\theta}$$

$$I_R = \lim_{R\to\infty} \int_0^{\pi} \frac{b_n(Re^{i\theta})^n + \ldots\ldots + b_0}{c_m(Re^{i\theta})^m + \ldots\ldots + c_n}\, iR\,d\theta$$

$$I_R = \lim_{R\to\infty} \int_0^{\pi} \frac{b_n}{c_m}(Re^{i\theta})^{n-m}\, iR\,d\theta$$

$$I_R = \lim_{R\to\infty} \int_0^{\pi} \frac{b_n}{c_m}(R)^{n+1-m}(e^{i\theta})^{n-m}\,d\theta$$

$$\epsilon \equiv -(n+1-m)$$

Then

$$\boxed{I_R = \lim_{R\to\infty} \frac{i}{R^{\epsilon}} \cdot \frac{b^n}{c_m} \int_0^{\pi} e^{i(n-m)\theta}\,d\theta}$$

Necessary and sufficient condition for $f(z)$ to be analytic

→ Necessary condition;

The necessary condition for a function $f(z) = u(x+y) + iv(x,y)$ to be analytic in domain D is that

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$z = x + iy$$
$$f(z) = u + iv$$
$$\Delta z = \Delta x + i\Delta y$$
$$z + \Delta z = (x+iy) + [(\Delta x) + (i\Delta y)]$$
$$= (x+\Delta x) + i(y+\Delta y)$$

$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u+iv]}{\Delta z}$$

case 1 –

$$= \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y) - u(x,y) + i[v(x+\Delta x, y)] - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \ldots\ldots (1)$$

case 2 ;

$$= \lim_{\Delta z \to 0} \frac{[u(x,y+\Delta y) - u(x,y)]}{i\Delta y} + i\left[\frac{v(x,y+\Delta y) - v(x,y)}{i\Delta y}\right]$$

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \ldots\ldots (2)$$

by eqⁿ (1) & (2)

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} \end{aligned}\right\} \text{ necessary condⁿ}$$

CR Equation

→ Sufficient condition –

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\left[\frac{\partial u}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y\right]+i\left[\frac{\partial v}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y\right]}{\Delta z}$$

$$= \frac{\frac{\partial u}{\partial x}(x+iy)+\frac{\partial v}{\partial x}(i\Delta x-\Delta y)}{\Delta z}$$

$$= \frac{\frac{\partial u}{\partial x}(\Delta x+i\Delta y)}{\Delta z} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$z \longrightarrow 0$

Then

$$\lim_{z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

• Residue Theorem –

An analytic $f(z)$ whose Laurent series is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

It can be integrated term by using a closed γ encircling z_0

$$\int_\gamma f(z)\,dz = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n dz$$

$$\int_\gamma f(z)dz = \sum_{n=-\infty}^{-2} a_n (z-z_0)^n dz + a_{-1}\int_\gamma \frac{1}{z-z_0}dz + \sum_{n=0}^{\infty} a_n \int_\gamma (z-z_0)^n dz$$

$$\int_\gamma f(z)\,dz = a_{-1}\int_\gamma \frac{1}{z-z_0}dz$$

↑ complex residue

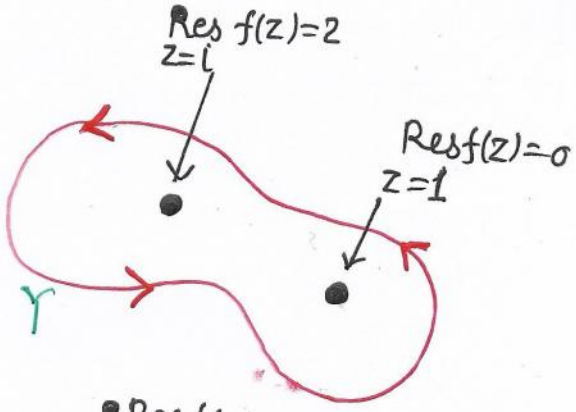
$$z = \gamma(t) = e^{it}+z_0$$

$$\int_\gamma \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{ie^{it}\,dt}{e^{it}} = 2\pi i$$

$$\int_\gamma f(z)\,dz = 2\pi i a_{-1}$$

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{a\in A} \operatorname*{Res}_{z=a_i} f(z)}$$

$\operatorname*{Res}_{z=3+2i} f(z) = i$



$\operatorname{Res} f(z) = -2$
$z=-i$

$\operatorname{Res} f(z) = 5$
$z=-1-2i$

$$f(z) = \frac{3}{(z-1)^2} + \frac{2}{z-i} - \frac{2}{z+i} + \frac{i}{z+3-2i} + \frac{5}{z+1+2i}$$

$$\int_\gamma f(z)\,dz = 2\pi i(0+2)$$

$$\boxed{\int_\gamma f(z)\,dz = 4\pi i}$$

UNIT-4
Matrix and Tensors

- **Matrix:**
  - **Definition –**
    A matrix is an $m \times n$ array of scalars from a given field F. The Individual values in the matrix are called entries

    e.g.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2 = m\times n}$$

    where m is number of rows
    n is number of columns

    Notation of matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

    Rows

    Column

    A = denotes of matrix
    a = entries of matrix where $a \in F$

  - **Types –**
    (1) Square matrix –
    A matrix having same no. of rows and columns is called square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

→ a_{11}, a_{21}, a_{33} are called diagonal matrix elements.
→ Diagonal elements of square matrix are called the trace of the matrix.

(ii) Diagonal matrix –

If all the elements of a square matrix are zero, except those in the leading diagonal then the matrix is said to be diagonal matrix.

e.g.
$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

(iii) Scalar Matrix –

A diagonal matrix in which all the diagonal elements are equal arecalled scalar matrix.

$$A = \begin{bmatrix} \lambda & 0 & 0 & .. & 0 \\ 0 & \lambda & 0 & .. & 0 \\ 0 & 0 & \lambda & .. & 0 \\ 0 & 0 & 0 & .. & \lambda \end{bmatrix}_{n \times n}$$

(iv) Identity or Unitary Matrix –

A scalar matrix in which each diagonal element is unitary.

e.g.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus on square matrix $[a_{ij}]$ is an unitary matrix if.

$$a_{ij} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$$

(v) Row matrix –

$$[x]_{1 \times n} = [a_{11}, a_{12} \dots a_{1n}]_{1 \times n}$$

(vi) Column matrix –

$$[x]_{m \times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ \dots \\ a_{m1} \end{bmatrix}_{m \times 1}$$

(vii) Null and Zero matrix-

If all the elements of matrix will be zero

$$[X]_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

- Inverse of a matrix-

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_{2\times2}$$

$$A^{-1} = ?$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1$$

$$|A| = 2 \neq 0$$

$$\text{adj } A = \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)' =$$

$$\text{adj } A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

- Trace of a matrix-

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 3 \\ 3 & 3 & 4 \end{bmatrix} = 1+4+4 = 9$$

- Orthogonality –

A matrix A is said to be orthogonal if $AA^T = I$

e.g.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A \cdot A^T = I}$$

- Eigen value and Eigen Vectors :

$$AX = \lambda X \quad \ldots\ldots (1)$$

$$A(X) = \lambda X$$

$$(A - \lambda I)X = 0$$

↑ It may be written as $(A - \lambda I)X = 0$

e.g. $$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$A-\lambda I=0$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} = 0$$

$\Rightarrow (2-\lambda)[(2-\lambda)(2-\lambda)-(1)]-(-1)[(-1)(2-\lambda)-(-1)(1)]+1[(-1)(-1)-2-\lambda(1)]=0$

$\Rightarrow (2-\lambda)[\lambda^2-4\lambda+3]+1[\lambda-1]+\lambda-1=0$

$\Rightarrow -\lambda^3-6\lambda^2-9\lambda+4=0$

$\Rightarrow (\lambda-4)(\lambda^2-2\lambda+1)=0$

$\Rightarrow (\lambda-4)(\lambda-1)(\lambda-1)=0$

$\boxed{\lambda=4,1,1}$

- Find Eigen Vectors-

$$A=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$(A-\lambda I)X=0$$

$\lambda_1=2$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$y=0$

$x+z=0 \Rightarrow 1-1=0$

$y=0$

$$x_i = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$|x_i| = \sqrt{(1)^2+(0)^2+(-1)^2}$$

$$|x_i| = \sqrt{2}$$

$$|\hat{x}_i| = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

same process for λ_2 and λ_3.

- Find adj A = ?

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\text{adj}A = \begin{bmatrix} +A_{11} & -A_{12} & +A_{13} \\ -A_{21} & +A_{22} & -A_{23} \\ +A_{31} & -A_{32} & +A_{33} \end{bmatrix}'$$

$$\text{adj}A = \begin{bmatrix} +(2-0) & -(2-0) & +(1-0) \\ -(0+1) & +(0-0) & -(-1-2) \\ +(0+2) & -(0-4) & +(0-2) \end{bmatrix}'$$

$$\text{adj}A = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 0 & 3 \\ 2 & 4 & -2 \end{bmatrix}'$$

$$\boxed{\text{adj}A = \begin{bmatrix} 2 & -1 & 2 \\ -2 & 0 & 4 \\ 1 & 3 & -2 \end{bmatrix}}$$

• Cayley Hamilton Theorem–

Every square matrix have its own characteristic eq^n } Statement
i.e. if for a square matrix $(n \times n)$ of order n.

The characteristic Polynomial is

$|A-\lambda I| = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n$

Then the matrix eq^n

$$a_0 + a_1 X + a_2 X^2 + \ldots + a_n X^n = 0$$

Proof;

The characteristic polynomial is

$|A-\lambda I| = a_0 + a_1\lambda^1 + a_2\lambda^2 + a_3\lambda^3 + \ldots + a_n\lambda^n \ldots\ldots\ldots (1)$

The characteristic of A is

$|A-\lambda I| = a_0 + a_1\lambda^1 + a_2\lambda^2 + a_3\lambda^3 + \ldots + a_n\lambda^n = 0 \ldots\ldots (2)$

The matrix eq^n is

$a_0 + a_1 X^1 + a_2 X^2 + \ldots + a_n X^n = 0 \ldots\ldots\ldots (3)$

If this eq^n is satisfied by A then we have to show that

$a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n = 0 \ldots\ldots\ldots (4)$

Since the each element of characteristic matrix $|A-\lambda I|$ is an ordinary polynomial of degree n.

The cofector of each and every element of $|A-\lambda I|$ is an ordinary polynomial of degree $n-1$, then;

$$B = \text{adj}\,|A-\lambda I|$$

$$B = \text{adj}\,|A-\lambda I| = B_0 + B_1\lambda^1 + B_2\lambda^2 + \ldots + B_{n-1}\lambda^{n-1} \ldots (5)$$

Where $B_0, B_1 \ldots B_{n-1}$ are the square matrix of same order n.

By eq^n(5) & (1)

$$|A-\lambda I|(B_0 + B_1\lambda^1 + B_2\lambda^2 + \ldots + B_{n-1}\lambda^{n-1}) = (a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 \ldots a_n\lambda^n) I$$

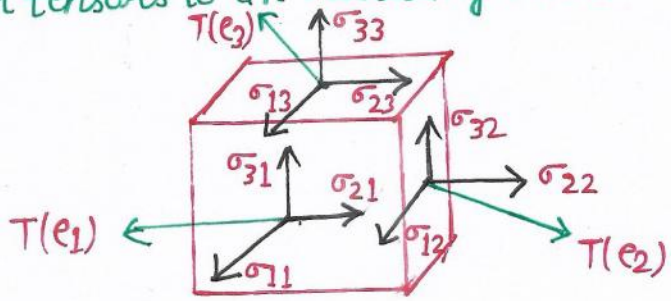
Compare the cofficient

$$AB_0 = a_0 I$$
$$AB_1 - B_0 = a_1 I$$
$$AB_2 - B_1 = a_2 I$$
$$\ldots\ldots\ldots,$$
$$AB_{n-1} - B_{n-2} = a_{n-1} I$$

$$a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n = 0$$

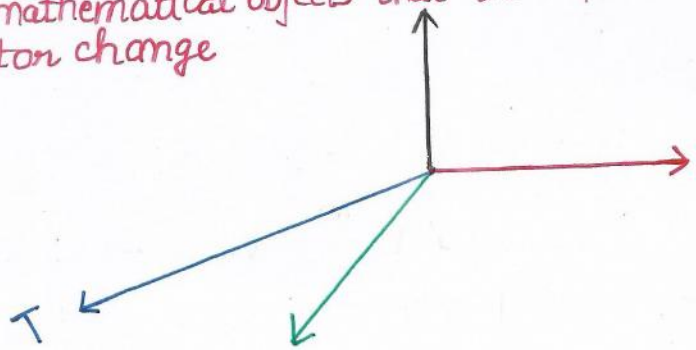
• Tensors-

In mathematics, a tensor is an arbitrary complex geomatric object that maps in a multilinear manner geomatric vectors and scalars and other tensors to an resulting tensor.



$$T = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$ → Cauchy second order tensor

Tensors are mathematical objects that transform in a special way when the basis vector change



• Multiplications of tensors-

$$\left\{ \begin{array}{l} g^{ij} A_j = A^i \\ g^{ij} A^j = A_i \end{array} \right\}$$ [Afriken 1985]

• contravarient Tensors-

A contravarient tensor is a tensor having specific transformation property.

$$dr = dx_1 \hat{x}_1 + dx_2 \hat{x}_2 + dx_3 \hat{x}_3$$

for which

$$dx_i' = \frac{\partial x_i'}{\partial x_j} dx_j$$

• Convarient Tensor-

A convarient tensor denoted with lowered index (e.g. a_{ij}) is a tensor having specific property

$$\nabla \phi = \frac{\partial \phi}{\partial x_1} \hat{x}_1 + \frac{\partial \phi}{\partial x_2} \hat{x}_2 + \frac{\partial \phi}{\partial x_3} \hat{x}_3 \text{ for which } \frac{\partial \phi'}{\partial x_i'} = \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial x_i'}$$