Integration Formula Sheet

Notes

Abhishek Bohare

a_bohare@icloud.com

typeset with LaTeX

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Todo list

In the notes the right-hand side is written as $\frac{\mathrm{d}}{\mathrm{d}x}\{g(x)\} + c = f(x)$; the usual statement is $\frac{\mathrm{d}}{\mathrm{d}x}\{g(x) + c\} = g'(x) = f(x)$. Kept as written. . . . 1

Items 14, 18, 20 and 26 differ from the standard results. Item 14: the last form should read $-\log(\operatorname{cosec} x + \cot x)$ (with a minus sign). Items 18 and 20 both give $\int \operatorname{cosech}^2 x \, \mathrm{d}x$ with opposite signs; the correct result is $-\coth x$ (item 18). Item 26: the logarithm should be $\log\left[x + \sqrt{x^2 - a^2}\right]$ (a plus sign), which equals $\cosh^{-1}(x/a)$ up to a constant. All are kept as written. 2

Item 2 shows a stray dash before $\mathrm{d}x$ in the scan; item 29 has an unclear first coefficient, read here as $x/2$; the label of item 33 is written "$\sin\cos bx$" and is read as $\cos bx$. 2

The second statement is written without limits; it presumably means $\int_a^a f(x)\,\mathrm{d}x = 0$. 4

P7, P8 and P9 hold only for a periodic function: f must have period a in P7 and P9, and period T in P8. The notes do not state the condition. 5

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1 Integral Calculus

These notes are a formula sheet for indefinite and definite integrals. Throughout, log means the natural logarithm, c is the constant of integration, and a, b, p, q are constants.

$$\int f(x)\,\mathrm{d}x = g(x) + c \iff \frac{\mathrm{d}}{\mathrm{d}x}\{g(x)\} + c = f(x)$$

In the notes the right-hand side is written as $\frac{\mathrm{d}}{\mathrm{d}x}\{g(x)\} + c = f(x)$; the usual statement is $\frac{\mathrm{d}}{\mathrm{d}x}\{g(x)+c\} = g'(x) = f(x)$. Kept as written.

1.1 Standard results

1. $\displaystyle\int (ax+b)^n\,\mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \quad n \neq -1$

2. $\displaystyle\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\log(ax+b) + c$

3. $\displaystyle\int e^{ax+b}\,\mathrm{d}x = \frac{1}{a}e^{ax+b} + c$

4. $\displaystyle\int a^{px+q}\,\mathrm{d}x = \frac{1}{p}\,\frac{a^{px+q}}{\log a} + c \quad (a > 0)$

5. $\displaystyle\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c$

6. $\displaystyle\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b) + c$

7. $\displaystyle\int \tan(ax+b)\,\mathrm{d}x = \frac{1}{a}\log\sec(ax+b) + c$

8. $\displaystyle\int \cot(ax+b)\,\mathrm{d}x = \frac{1}{a}\log\sin(ax+b) + c$

9. $\displaystyle\int \sec^2(ax+b)\,\mathrm{d}x = \frac{1}{a}\tan(ax+b) + c$

10. $\displaystyle\int \operatorname{cosec}^2(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cot(ax+b) + c$

11. $\displaystyle\int \sec(ax+b)\tan(ax+b)\,\mathrm{d}x = \frac{1}{a}\sec(ax+b) + c$

12. $\displaystyle\int \operatorname{cosec}(ax+b)\cot(ax+b)\,\mathrm{d}x = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$

13. $\displaystyle\int \sec x\,\mathrm{d}x = \log(\sec x + \tan x) + c = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

14. $\int \operatorname{cosec} x \, \mathrm{d}x = \log(\operatorname{cosec} x - \cot x) + c = \log \tan \frac{x}{2} + c = \log(\operatorname{cosec} x + \cot x) + c$

15. $\int \sinh x \, \mathrm{d}x = \cosh x + c$

16. $\int \cosh x \, \mathrm{d}x = \sinh x + c$

17. $\int \operatorname{sech}^2 x \, \mathrm{d}x = \tanh x + c$

18. $\int \operatorname{cosech}^2 x \, \mathrm{d}x = -\coth x + c$

19. $\int \operatorname{sech} x \tanh x \, \mathrm{d}x = -\operatorname{sech} x + c$

20. $\int \operatorname{cosech}^2 x \, \mathrm{d}x = \coth x + c$

21. $\int \operatorname{cosech} x \coth x \, \mathrm{d}x = -\operatorname{cosech} x + c$

22. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$

23. $\displaystyle\int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

24. $\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$

25. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2 + a^2}} = \log\left[x + \sqrt{x^2 + a^2}\right] + c \quad \text{or} \quad \sinh^{-1} \frac{x}{a} + c$

26. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2 - a^2}} = \log\left[x - \sqrt{x^2 - a^2}\right] + c \quad \text{or} \quad \cosh^{-1} \frac{x}{a} + c$

27. $\displaystyle\int \frac{\mathrm{d}x}{a^2 - x^2} = \frac{1}{2a} \log \frac{a + x}{a - x} + c$

28. $\displaystyle\int \frac{\mathrm{d}x}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a} + c$

29. $\displaystyle\int \sqrt{a^2 - x^2} \, \mathrm{d}x = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

30. $\displaystyle\int \sqrt{x^2 + a^2} \, \mathrm{d}x = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + c$

31. $\displaystyle\int \sqrt{x^2 - a^2} \, \mathrm{d}x = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a} + c$

32. $\displaystyle\int e^{ax} \sin bx \, \mathrm{d}x = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$

33. $\displaystyle\int e^{ax} \cos bx \, \mathrm{d}x = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c$

Items 14, 18, 20 and 26 differ from the standard results. Item 14: the last form should read $-\log(\operatorname{cosec} x + \cot x)$ (with a minus sign). Items 18 and 20 both give $\int \operatorname{cosech}^2 x \, \mathrm{d}x$ with opposite signs; the correct result is $-\coth x$ (item 18). Item 26: the logarithm should

1.2 Techniques of integration

1.2.1 Substitution

$$I = \int f(x)\,\mathrm{d}x = \int f(\phi(t))\,\phi'(t)\,\mathrm{d}t, \qquad x = \phi(t).$$

1.2.2 Integration by parts

$$\int u\,v\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int \left[\frac{\mathrm{d}u}{\mathrm{d}x}\int v\,\mathrm{d}x\right]\mathrm{d}x.$$

The order in which to choose u is given by the rule ILATE:

I inverse function; **L** logarithmic function; **A** algebraic function; **T** trigonometric function; **E** exponential function.

2 Definite Integral

2.1 Basic result

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a), \qquad \text{where } \int f(x)\,\mathrm{d}x = F(x) + c.$$

$$\Rightarrow \int f(x)\,\mathrm{d}x = 0 \text{ if } f \text{ is a continuous function in } (a, b).$$

The second statement is written without limits; it presumably means $\int_a^a f(x)\,\mathrm{d}x = 0$.

2.2 Properties

P1. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t$

P2. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

P3. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$

P4. $\displaystyle\int_{-a}^a f(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } f(x) \text{ is odd, } f(x) = -f(-x), \\ 2\displaystyle\int_0^a f(x)\,\mathrm{d}x & \text{if } f(x) \text{ is even, } f(x) = f(-x). \end{cases}$

P5. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(a+b-x)\,\mathrm{d}x$. In particular, $\displaystyle\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x$.

P6. $\displaystyle\int_0^{2a} f(x)\,\mathrm{d}x = \int_0^a f(x)\,\mathrm{d}x + \int_0^a f(2a-x)\,\mathrm{d}x = \begin{cases} 2\displaystyle\int_0^a f(x)\,\mathrm{d}x & \text{if } f(2a-x) = f(x), \\ 0 & \text{if } f(2a-x) = -f(x). \end{cases}$

P7. $\displaystyle\int_0^{na} f(x)\,\mathrm{d}x = n\int_0^a f(x)\,\mathrm{d}x$

P8. $\displaystyle\int_{a+nT}^{b+nT} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$

P9. $\displaystyle\int_{ma}^{na} f(x)\,\mathrm{d}x = (n-m)\int_0^a f(x)\,\mathrm{d}x$

P10. If $f(x) \le \phi(x)$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \le \int_a^b \phi(x)\,\mathrm{d}x$.

P11. $\displaystyle\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x$

P12. If $f(x) \geq 0$ on the interval $[a, b]$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$.

P7, P8 and P9 hold only for a periodic function: f must have period a in P7 and P9, and period T in P8. The notes do not state the condition.

2.3 Wallis's formula

$$\int_0^{\pi/2} \sin^n x \cos^m x\,\mathrm{d}x = \frac{[(n-1)(n-3)(n-5)\cdots][(m-1)(m-3)\cdots]}{(m+n)(m+n-2)(m+n-4)\cdots} K = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)},$$

where $K = \dfrac{\pi}{2}$ when m and n are both even, and $K = 1$ otherwise.

The last factors of each product in the notes are hard to read (they look like "$\log 2$", and are read here as the usual end of the product, 2 or 1). The condition on K is written "$K = \pi/2$ if even, $= 1$ if odd"; the usual statement is that $K = \pi/2$ when m and n are both even.

2.4 Derivative of an anti-derivative function

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)\,\mathrm{d}t = f[h(x)]\,h'(x) - f[g(x)]\,g'(x).$$

2.5 Definite integral as the limit of a sum

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} h[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)] = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh),$$

where $b - a = nh$. If $a = 0$ and $b = 1$, then $nh = 1$ and

$$\lim_{n\to\infty} h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)\,\mathrm{d}x, \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,\mathrm{d}x.$$

In the last formula the sum runs from $r = 1$ to $n - 1$ in the notes; the usual form uses $r = 0$ to $n - 1$ or $r = 1$ to n. The limit is the same in each case.

2.6 Estimation of a definite integral

(i) For a monotonically *decreasing* function on (a, b):

$$f(b)\,(b-a) < \int_a^b f(x)\,\mathrm{d}x < f(a)\,(b-a).$$

(ii) For a monotonically *increasing* function on (a, b):

$$f(a)\,(b-a) < \int_a^b f(x)\,\mathrm{d}x < f(b)\,(b-a).$$

2.7 Series expansions

1. $\log 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{5} - \cdots$

2. $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots + \infty$

3. $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \infty$

4. $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \infty$

5. $\frac{\pi^2}{24} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots \infty$